Convergence and stability of estimated error variances derived from assimilation residuals in observation space

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Abstract
The convergence of the Desrosiers’ scheme to estimate observation and background error variances based on Omitr, Omitb and Anmt is studied from a theoretical point of view. The general properties of the fixed point of the scheme are discussed and illustrated with a scalar, 1D domain, and in an operational assimilation system. Several iterated schemes are considered: the estimation of either observation or background error variances and the estimation of both variances either simultaneously or in sequence.

It is shown that for the simultaneous estimation of observation and background error variance the theoretical convergence is obtained in a single iteration, but the convergent value are incorrect although the sum of variances matches the innovation variance. Additional information (e.g., correlation model, lagged-innovation) is needed to resolve the estimation problem.

Convergence – scalar case

- Iteration on observation error

\[ \hat{E}(O - A(O - F)^T) = HBR^2 + \hat{R} \]

where \( \hat{O} = (O - A(O - F)^T) \) is obtained from assimilation residuals and overlap occurs predefined error covariances

- Correctly prescribed forecast error variances

\[ \hat{E}(O - A(O - F)^T) = HBR^2 + \hat{R} \]

where \( \hat{O} = (O - A(O - F)^T) \) is the next iterate

- so the iteration on \( \hat{O} \) takes the form

\[ \hat{O} = \frac{\hat{O}}{1 + \hat{R} \hat{R}} + \hat{G}(\hat{O}) \]

Define a mapping \( \hat{G}(\hat{O}) \)

\[ \hat{G}(\hat{O}) = \frac{\hat{O}}{1 + \hat{R} \hat{R}} + \hat{G}(\hat{O}) \]

The fixed point is

\[ \hat{O} = \hat{G}(\hat{O}) \]

condition for convergence

\[ \frac{\hat{O}}{1 + \hat{R} \hat{R}} = 1 \]

and so for this case we get \( \hat{O}^2 = 1 \)

\[ \hat{G}(\hat{O})^2 = \frac{1}{\hat{O}^2 + \hat{R} \hat{R}} + \hat{G}(\hat{O}) \]

the scheme is always convergent and converges to the true value \( \hat{O} = 1 \)

- Incorrectly prescribed forecast error variance

\[ \hat{E}(O - A(O - F)^T) = HBR^2 + \hat{R} \]

The fixed point is

\[ \hat{O} = \frac{\hat{O}}{1 + \hat{R} \hat{R}} + \hat{G}(\hat{O}) \]

that is not the true observation error value

Convergence – 1D domain – Simultaneous

Case where the background error variance is specified, covariably and the observation error covariance is quasianalytical

Assume an homogeneous \( B \) in a 1D periodic domain with observations at each grid points, \( \hat{H} = 1 \)

We can write the Fourier transform as a matrix \( \hat{F} \) and its inverse as \( \hat{F}^T \)

Then the system

\[ \hat{R}_x = \hat{R}_b \hat{B} \]

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All matrices can be simultaneously diagonalized giving 3 systems of scalar (variance) equations (one for each wavenumber k)

\[ \hat{R}_x = \hat{R}_b \hat{B} \]

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It will not converge \( \hat{R}_x \rightarrow \hat{R}_b \)

Please provide the Innovation variance, as for any practical cases the scheme converges.

Summary and Conclusions
- The convergence of the Desrosiers’ et al (2005) scheme has been investigated from a theoretical context and from an assimilation cycle
- Iteration on either observation error variance or background error variance generally converges, but will converge to an overestimate if the counterpart in underestimated, and vice versa
- Iteration on both observation and background error variance converges in a single step, to a non-unique fixed-point solution. On the fixed point solution the innovation variance of the solution is the same as the innovation variance
- An analysis in a 1D domain reveals the same behavior. While the spectral variance of the estimated innovation matches that of the spectral innovation variance, the individual component, i.e. the observation error and background error do not converge to the truth. In particular, the observation error becomes spatially correlated and the background error variance spectrum becomes more red.

An Iterated map \( G \)

\[ x_{n+1} = G(x_n) \]

has a fixed point \( x^* = G(x^*) \)

The scheme is convergent if

\[ \frac{\partial G(x^*)}{\partial x} < 1 \]

General properties of the Desrosiers’ scheme

(1) If \( \hat{R}_x \) and \( \hat{F}_b \) are fixed points, then

\[ \hat{R}_b \hat{F}_b = \hat{R}_x \]

(2) When the iterate \( k \) is such that

\[ \hat{R}_b \hat{F}_b = \hat{R}_x \]

no more updates on the individual components \( \hat{R}_x \) and \( \hat{B}_b \) can occur.